

A note on eigenvalues of random block Toeplitz matrices with slowly growing bandwidth

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Abstract

This paper can be thought of as a remark of [5], where the authors studied the eigenvalue distribution μ_{X_N} of random block Toeplitz band matrices with given block order m . In this note we will give explicit density functions of $\lim_{N \rightarrow \infty} \mu_{X_N}$ when the bandwidth grows slowly. In fact, these densities are exactly the normalized one-point correlation functions of $m \times m$ Gaussian unitary ensemble (GUE for short). The series $\{\lim_{N \rightarrow \infty} \mu_{X_N} | m \in \mathbb{N}\}$ can be seen as a transition from the standard normal distribution to semicircle distribution. We also show a similar relationship between GOE and block Toeplitz band matrices with symmetric blocks.

Keywords: block Toeplitz matrix, GUE, GOE, limit spectral distribution

1 Introduction

A block Toeplitz matrix is a block matrix which can be written as

$$T_N = (A_{i-j})_{i,j=1}^N = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots & A_{-(N-1)} \\ A_1 & A_0 & A_{-1} & \cdots & A_{-(N-2)} \\ A_2 & A_1 & A_0 & \cdots & A_{-(N-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{N-1} & A_{N-2} & A_{N-3} & \cdots & A_0 \end{pmatrix}$$

where $\{A_{-(N-1)}, \dots, A_0, \dots, A_{N-1}\}$ are $m \times m$ matrices and $A_s = (a_{ij}(s))_{i,j=1}^m$. If the $a_{ij}(s)$'s are random variables, then we call T_N a random block Toeplitz matrix. We suppose the $a_{ij}(s)$'s are real random variables and $A_s = (A_{-s})^T$. Besides, we list some assumptions as follows.

Independence of the elements:

- (1) $a_{i_1 j_1}(s_1)$ and $a_{i_2 j_2}(s_2)$ are independent if $|s_1| \neq |s_2|$,

- (2) If $s \neq 0$ and $(i_1, j_1) \neq (i_2, j_2)$ then $a_{i_1 j_1}(s)$ and $a_{i_2 j_2}(s)$ are independent,
(3) If $(i_1, j_1) \neq (i_2, j_2)$ and $(i_1, j_1) \neq (j_2, i_2)$ then $a_{i_1 j_1}(0)$ and $a_{i_2 j_2}(0)$ are independent.

Uniform boundness condition:

- (4) $\mathbb{E}[a_{ij}(s)] = 0$, $\mathbb{E}[|a_{ij}(s)|^2] = 1$, $-(N-1) \leq s \leq N-1$, $1 \leq i, j \leq m$
and

$$\sup_{\substack{N \in \mathbb{N} \\ -(N-1) \leq s \leq N-1}} \{\mathbb{E}[|a_{ij}(s)|^k] | 1 \leq i, j \leq m\} = C_{k,m} < +\infty$$

and

$$\sup_{m \in \mathbb{N}} C_{k,m} = C_k < +\infty.$$

Slowly growing bandwidth condition:

- (5) The block Toeplitz matrix is a band block matrix with bandwidth b_N , that is, $A_s = 0$ for $|s| > b_N$. Moreover, b_N satisfies: $\lim_{N \rightarrow \infty} b_N = \infty$ and $b_N = o(N)$.

Gaussian unitary ensemble (GUE for short) $(\mathcal{H}_m, d\mu)$ is the space \mathcal{H}_m of all Hermitian $(m \times m)$ -matrices with a certain Gaussian measure $d\mu$ (see [1, 6]). We will regularly use the notations $\langle \cdot \rangle_{\text{GUE}}$ and $\langle \cdot \rangle_{\text{TBM}}$, respectively denoting the expectations under GUE and random Toeplitz band matrices. If $H = (h_{ij})$ is a matrix from GUE, then it is easy to see $\langle h_{ij} h_{ji} \rangle_{\text{GUE}} = 1$ while all other second moments are equal to zero: $\langle h_{ij} h_{kl} \rangle_{\text{GUE}} = 0$, whenever $(i, j) \neq (l, k)$.

If $A = (a_{ij}(\omega))_{i,j=1}^N$ is a real symmetric or complex Hermitian random matrix and its entries are random variables on a probability space Ω with a probability measure P , then the eigenvalue distribution of A is

$$\mu_A = \frac{1}{N} \int_{\Omega} \sum_{j=1}^N \delta_{\lambda_j(\omega)} dP(\omega)$$

where $\lambda_j(\omega)$'s are the N real eigenvalues of A . We have

Theorem 1.1. *Let T_N be an $mN \times mN$ random block Toeplitz matrix as above. Set $X_N = \frac{T_N}{\sqrt{2mb_N}}$, then μ_{X_N} converges weakly to $f_m(x)dx$ as $N \rightarrow \infty$.*

Moreover, if the bandwidth satisfies $\sum_{N=1}^{\infty} b_N^{-2} < \infty$, then the convergence is almost sure. Here $f_m(x) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \psi_j^2(\sqrt{m}x)$ where ψ_j is the j th normalized oscillator wave-function:

$\psi_j(x) = \frac{e^{-\frac{x^2}{4}} H_j(x)}{\sqrt{\sqrt{2\pi} j!}}$ and H_j is the j th Hermite polynomial: $H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{-\frac{x^2}{2}}$.

Remark. $f_m(x)$ is the one-point correlation function of GUE up to the scaling [6]. As f_1 is the density function of the standard normal distribution and when $m \rightarrow \infty$ f_m converges in law to the semicircle distribution (see [5]), $\{\lim_{N \rightarrow \infty} \mu_{X_N} | m \in \mathbb{N}\}$ can be seen as a transition from $N(0, 1)$ to semicircle distribution.

In [4], Kologlu, Kopp and Miller got a very similar result. They proved that the limiting eigenvalue density of symmetric block circulant Toeplitz ensemble

is the same as the eigenvalue density of GUE (Theorem 1.4(1) of [4]). But there is an essential difference between their model and our model: In [4] the block Toeplitz matrix has a circulant structure while in this paper the block Toeplitz matrix has a band structure. In the viewpoint of combinatorics, both of the two structures are strong. So they are two different models induced from the block Toeplitz matrix model and interestingly have the same limiting eigenvalue distribution. There is also another difference: In [4] the entries of the block Toeplitz matrix are i.i.d random variables but in our model the entries only have to satisfy the independent condition.

Theorem 1.2. *Let ν_1, \dots, ν_r be nonnegative integers, and $\nu = \nu_1 + \dots + \nu_r$. Set $Y_N = \frac{T_N}{\sqrt{2b_N}}$, then*

$$\lim_{N \rightarrow \infty} \frac{1}{N^\nu} \left\langle \prod_{i=1}^r (\text{tr} Y_N^i)^{\nu_i} \right\rangle_{\text{TBM}} = \left\langle \prod_{i=1}^r (\text{tr} H^i)^{\nu_i} \right\rangle_{\text{GUE}}. \quad (1.1)$$

2 Proof of Main Results

Definition 2.1. Let $[n] = \{1, 2, \dots, n\}$, $\forall n \in \mathbb{N}$.

(1) We call $\pi = \{\{a_1, b_1\}, \dots, \{a_k, b_k\}\}$ a pair partition of $[2k]$ if $\bigcup_{j=1}^r \{a_j, b_j\} = [2k]$

and $\{a_i, b_i\} \cap \{a_j, b_j\} = \emptyset$ if $i \neq j$. For convenience, we assume that $a_1 < \dots < a_k$ and $a_i < b_i$ for each pair, under which such a pair partition is called a Wick coupling. For π , we define $\pi(a_j) = b_j$ and $\pi(b_j) = a_j$ ($1 \leq j \leq k$). We denote by $\mathcal{P}_2(2k)$ the set of pair partitions of $[2k]$.

(2) Suppose $\pi \in \mathcal{P}_2(2k)$. Then π can be seen as a permutation: $(a_1, b_1) \cdots (a_k, b_k)$. Consider the canonical cycle $\gamma_0 = (1, 2, \dots, 2k-1, 2k)$. Let $g(\pi)$ denote the number of orbits of the permutation $\gamma_0 \circ \pi$.

The following lemma is a well-known consequence of Wick's formula on the moments of GUE. One can get it with the method of moment generating function (see Section 3.3.1 of [1]).

Lemma 2.2. *Suppose $H = (b_{ij})_{i,j=1}^m$ is an $m \times m$ random Hermitian matrix from GUE. Set $Y = \frac{H}{\sqrt{m}}$. Then the odd moments of μ_Y are all 0 and the even moments of μ_Y are $m_{2k}(\mu_Y) = \sum_{\pi \in \mathcal{P}_2(2k)} m^{g(\pi)-k-1}$.*

Proof of Theorem 1.1. From Theorem 4.3 of [5] and Lemma 2.2, we know that μ_Y and γ_T^m have the same moments. It follows from Carleman's Theorem (see [3]) that γ_T^m should be μ_Y . So the density function of γ_T^m is the one-point correlation function of GUE, which is a well-known function [1, 6]:

$$f_m(x) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \psi_j^2(\sqrt{m}x).$$

From [2] we know if $\sum_{N=1}^{\infty} b_N^{-2} < \infty$, then the convergence is almost sure. \square

Proof of Theorem 1.2. By Lemma 2.2 of [5], the main contribution in the expansion of $\frac{1}{N^\nu} \langle \prod_{i=1}^r (\text{tr} Y_N^i)^{\nu_i} \rangle_{\text{TBM}}$ comes from the pair partitions. As in the proof of Theorem 4.3 in [5], we can show that

$$\lim_{N \rightarrow \infty} \frac{1}{N^\nu} \langle \prod_{i=1}^r (\text{tr} Y_N^i)^{\nu_i} \rangle_{\text{TBM}} = \begin{cases} 0 & \text{if } \sum_{i=1}^r i\nu_i \text{ is odd} \\ \sum_{\pi \in \mathcal{P}_2(\sum_{i=1}^r i\nu_i)} m^{F(\pi)} & \text{if } \sum_{i=1}^r i\nu_i \text{ is even} \end{cases}.$$

Here the definition of $F(\pi)$ as follows: Suppose that $\sum_{i=1}^r i\nu_i = 2\tilde{\nu}$ is even and $\pi = \{\{a_1, b_1\}, \dots, \{a_{\tilde{\nu}}, b_{\tilde{\nu}}\}\} \in \mathcal{P}_2(2\tilde{\nu})$, then $F(\pi)$ denotes the number of “free indices” of the system

$$\begin{cases} t_{a_i} = t_{f(b_i)} \\ t_{b_i} = t_{f(a_i)} \end{cases} \quad (1 \leq i \leq \tilde{\nu}), \text{ where } f \text{ is defined as below:}$$

Set $\nu_0 = 0$. For $1 \leq x \leq \sum_{i=1}^r i\nu_i$, if $\exists 0 \leq s \leq r-1$ and $1 \leq a \leq \nu_{s+1}$ such

that $x = \sum_{i=0}^s i\nu_i + (s+1)a$, then $f(x) = \sum_{i=0}^s i\nu_i + (s+1)(a-1) + 1$; otherwise $f(x) = x + 1$.

By Wick's formula, we can compute the integral $\langle \prod_{i=1}^r (\text{tr} H^i)^{\nu_i} \rangle_{\text{GUE}}$ and complete the proof. \square

3 Similar Results for GOE and Block Toeplitz Matrix with Symmetric Blocks

First, we remark that we can get the same results associated with GUE if one of the following conditions are imposed:

- 1) each block of T_N is a complex matrix and $A_{-s} = (\overline{A_s})^T$;
- 2) each block of T_N is a Hermitian matrix, that is, $A_{-s} = A_s = (\overline{A_s})^T$.

However, the situation becomes different when each block of $T_N = (A_{i-j})_{i,j=1}^N$ is a symmetric matrix, that is, $A_{-s} = A_s = (A_s)^T$. Besides, we modify the second moments of each block as follows: $\mathbb{E}[|a_{ij}(s)|^2] = \begin{cases} 1 & \text{if } i \neq j \\ 2 & \text{if } i = j \end{cases}, \forall s$.

We still use the notations $\langle \cdot \rangle_{\text{GOE}}$ and $\langle \cdot \rangle_{\text{TBM}}$, respectively denoting the expectations under GOE and Toeplitz band matrices with symmetric blocks.

Theorem 3.1. *Let T_N be an $mN \times mN$ random block Toeplitz matrix as mentioned above. Set $X_N = \frac{T_N}{\sqrt{2mb_N}}$, then*

$$\lim_{N \rightarrow \infty} \frac{1}{mN} \langle \text{tr} X_N^k \rangle_{\text{TBM}} = \frac{1}{m} \langle \text{tr} (H/\sqrt{m})^k \rangle_{\text{GOE}}. \quad (3.2)$$

Moreover, μ_{X_N} converges weakly to $g_m(x)dx$ as $N \rightarrow \infty$. And if the bandwidth satisfies $\sum_{N=1}^{\infty} b_N^{-2} < \infty$, then the convergence is almost sure. Here

$$g_m(x) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} \psi_j^2(\sqrt{m}x) + \left(\frac{m}{2}\right)^{1/2} \psi_{m-1}(\sqrt{m}x) \int_{-\infty}^{\infty} \varepsilon(x-t) \psi_m(\sqrt{m}t) dt + \alpha_m(x), \quad (3.3)$$

$\varepsilon(x) = \frac{1}{2} \text{sign}(x)$ and ψ_j is the j th nomarlized oscillator wave-function as in Theorem 1.1, while

$$\alpha_m(x) = \begin{cases} \frac{1}{m} \psi_{2s}(\sqrt{m}x) \div \int_{-\infty}^{\infty} \psi_{2s}(\sqrt{m}t) dt & \text{if } m = 2s + 1 \\ 0 & \text{if } m = 2s \end{cases}.$$

Proof of Theorem 3.1. For $H = (h_{ij})_{i,j=1}^m$, let $Y = \frac{H}{\sqrt{m}}$. From Wick's formula we know $\frac{1}{m} \langle \text{tr}(H/\sqrt{m})^k \rangle_{\text{GOE}} = 0$ when k is odd and $\frac{1}{m} \langle \text{tr}(H/\sqrt{m})^{2k} \rangle_{\text{GOE}}$ is

$$m^{-k-1} \sum_{t_1, \dots, t_{2k}=1}^m \sum_{\pi} (\langle h_{t_{a_1} t_{a_1+1}} h_{t_{b_1} t_{b_1+1}} \rangle_{\text{GOE}} \cdots \langle h_{t_{a_k} t_{a_k+1}} h_{t_{b_k} t_{b_k+1}} \rangle_{\text{GOE}})$$

where the second sum is taken over all $\pi = \{\{a_1, b_1\}, \dots, \{a_k, b_k\}\} \in \mathcal{P}_2(2k)$ and $t_{2k+1} := t_1$. H is symmetric, so $\langle h_{t_{a_1} t_{a_1+1}} h_{t_{b_1} t_{b_1+1}} \rangle_{\text{GOE}} \cdots \langle h_{t_{a_k} t_{a_k+1}} h_{t_{b_k} t_{b_k+1}} \rangle_{\text{GOE}} \neq 0$ implies

$$\begin{cases} t_{a_i} = t_{b_i+1} \\ t_{b_i} = t_{a_i+1} \end{cases} \quad \text{or} \quad \begin{cases} t_{a_i} = t_{b_i} \\ t_{a_i+1} = t_{b_i+1} \end{cases} \quad \text{for } 1 \leq i \leq k, \text{ moreover, each term}$$

$$\langle h_{t_{a_i} t_{a_i+1}} h_{t_{b_i} t_{b_i+1}} \rangle_{\text{GUE}} = \begin{cases} 2 & \text{if } t_{a_i} = t_{b_i} = t_{a_i+1} = t_{b_i+1} \\ 1 & \text{otherwise} \end{cases}$$

For a given $\pi = \{\{a_1, b_1\}, \dots, \{a_k, b_k\}\} \in \mathcal{P}_2(2k)$, set

$$A(\pi) = \left\{ (t_1, \dots, t_{2k}) \in [m]^{2k} \mid \begin{cases} t_{a_i} = t_{b_i+1} \\ t_{b_i} = t_{a_i+1} \end{cases} \quad \text{or} \quad \begin{cases} t_{a_i} = t_{b_i} \\ t_{a_i+1} = t_{b_i+1} \end{cases} \quad \text{for } 1 \leq i \leq k \right\}.$$

For $\mathbf{t} = (t_1, \dots, t_{2k}) \in A(\pi)$, let $r(\pi, \mathbf{t}) = \#\{i \in [k] \mid t_{a_i} = t_{b_i} = t_{a_i+1} = t_{b_i+1}\}$, then $\frac{1}{m} \langle \text{tr}(H/\sqrt{m})^{2k} \rangle_{\text{GOE}} = m^{-k-1} \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{\mathbf{t} \in A(\pi)} 2^{r(\pi, \mathbf{t})}$.

Similarly as in the proof of Theorem 3.2 and Theorem 4.3 of [5], we know when k is odd $\frac{1}{mN} \langle \text{tr} X_N^k \rangle_{\text{TBM}} = o(1)$ and $\frac{1}{mN} \langle \text{tr} X_N^{2k} \rangle_{\text{TBM}}$ is

$$\begin{aligned} & \sum_{i=1}^N \sum_{j_1, \dots, j_{2k} = -b_N}^{b_N} \sum_{t_1, \dots, t_{2k}=1}^m \frac{E(a_{t_1 t_2}(j_1) \cdots a_{t_{2k} t_1}(j_{2k}))}{(2mb_N)^k \cdot mN} \prod_{l=1}^{2k} I_{[1, N]}(i + \sum_{q=1}^l j_q) \delta_{0, \sum_{q=1}^{2k} j_q} \\ &= \sum_{i=1}^N \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{\mathbf{t} \in A(\pi)} \frac{2^{r(\pi, \mathbf{t})}}{(2mb_N)^k \cdot mN} \sum_{x_1, \dots, x_k = -b_N}^{b_N} \prod_{l=1}^{2k} I_{[1, N]}(i + \sum_{q=1}^l \epsilon_{\pi}(q) x_{\pi(q)}) + o(1) \\ &\rightarrow \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{\mathbf{t} \in A(\pi)} \frac{2^{r(\pi, \mathbf{t})}}{m^{k+1}} (N \rightarrow \infty). \end{aligned}$$

Thus $\lim_{N \rightarrow \infty} \frac{1}{mN} \langle \text{tr} X_N^k \rangle_{\text{TBM}} = \frac{1}{m} \langle \text{tr} (H/\sqrt{m})^k \rangle_{\text{GOE}}$. From [2] we know the convergence is almost sure if $\sum_{N=1}^{\infty} b_N^{-2} < \infty$. Finally, $g_m(x)$ is the one-point correlation function of GOE with order m , thus we complete the proof. \square

Remark. The density (3.3) follows from the 1-point correlation function of GOE (see (7.2.32) of [6]). This family densities can also be seen as a transition from the normal distribution $N(0, 2)$ to the semicircle distribution with variance 1. In the situation of GOE, we also have parallel results to Theorem 1.2.

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